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Synthetic Braided Geometry II

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Braided geometry is a natural generalization of supergeometry and is intimately connected with noncommutative geometry. Synthetic differential geometry is a peppy dissident in the stale regime of orthodox differential geometry, just as Grothendieck's category-theoretic revolution in algebraic geometry was in the middle of the 20th century. Our previous paper [Nishimura (1998) International Journal of Theoretical Physics, **37**, 2833–2849] was a gambit of our ambitious plan to approach braided geometry from a synthetic viewpoint and to concoct what is supposedly to be called *synthetic braided geometry*. As its sequel this paper is intended to give a synthetic treatment of braided connections, in which the second Bianchi identity is established. Considerations are confined to the case that the braided monoidal category at issue is a category of vector spaces graded by a finite Abelian group with its *nonsymmetric* braiding being given by phase factors. Thus the present paper encompasses physical systems amenable to *anyonic* statistics.

0. INTRODUCTION

Synthetic differential geometry is the vanguard of modern differential geometry, in which nilpotent infinitesimals are not only abundantly available as in the age of Riemann, Lie and Cartan, but also coherently organized with mathematical rigor. Synthetic differential geometry was pioneered by Lawvere, a famous category-theorist, in the middle of the 1960's, while Grothendieck revolutionized algebraic geometry by exploiting ideas of category theory (e.g., representable functors). Although Grothendieck's category-theoretic revolution in algebraic geometry during the middle of the 20th century is well appreciated among contemporary algebraic geometers, Lawvere's corresponding one in differential geometry has not received more than studied indifference from orthodox differential geometers. The so-called tensor analysis on infinitesimal entities (e.g., vector fields) in orthodox differential geometry is often stodgy and factitious, concealing the truly infinitesimal nature of infinitesimal considerations under a topsy-turvy of lengthy calculations in a dull drone. Synthetic differential geometry enables us to endow differential geometry with an infinitesimal horizon relatively independent of local

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and global ones. The standard notion of smooth manifold in orthodox differential geometry is not susceptible of any reasonable or fruitful generalization to noncommutative or braided geometry, while the central notion of microlinear space in synthetic differential geometry is. For good textbooks on synthetic differential geometry, the reader is referred to Lavendhomme (1996) (devoted mainly to a consistently axiomatic presentation of synthetic differential geometry) and Moerdijk and Reyes (1991) (devoted to model theory of synthetic differential geometry) as well as to Kock's bible of the field (Kock, 1981). We assume the reader to be familiar with Lavendhomme (1996) up to Chapter 5.

Supergeometry enables us to deal with bosons and fermions on an equal footing by intermingling them. It is expected to play a central role in any possible unification of relativity and quantum theory. Supergeometry lies at the entrance to noncommutative geometry in the sense that the ring of real supernumbers is not commutative but graded-commutative. For good textbooks on supergeometry the reader is referred to Bartoci *et al.* (1991), Leites (1980), and Manin (1988). We have approached supergeometry from a synthetic viewpoint in Nishimura (1998a, 1999, 2000a,c). In particular, a synthetic treatment of superconnections was given in Nishimura (2000a).

Braided geometry is an elegant and far-reaching generalization of supergeometry, in which the category of vector spaces is replaced by a braided monoidal category. It is pioneered and championed by Majid (1995a,b), Marcinek (1994), and others. The standard gadget for transmogrifying braided geometry into noncommutative geometry is bosonization, while the standard device for translating noncommutative geometry into braided geometry is transmutation. If the braiding is symmetric, braided geometry lies in the very periphery of supergeometry so that it is not truely braided. We will consider a (nonsymmetric) braided monoidal category of vector spaces graded by a finite Abelian group whose braiding is given by a phase factor. Our considerations do not only encompass color geometry but also anyonic geometry. For anyonic geometry the reader is referred to Majid (1993, 1994, 1997).

As a sequel to Nishimura (1998b) this paper gives a synthetic treatment of braided connections by generalizing our synthetic treatment of superconnections in Nishimura (2000a). Basic definitions and basic properties will be presented in Section 3. Section 5 is devoted to a combinatorial treatment of the so-called second Bianchi identity. Sections 4 and 6 are devoted to induced braided connections. We have gathered some preliminaries in Section 1. We deal in haste with braided exterior differential calculus in Section 2.

As is usual in synthetic differential geometry, the reader should presume throughout the paper that we are working in a (not necessarily Boolean) topos, so that the excluded middle and Zorn's lemma have to be avoided. Objects of the topos go under such aliases as a "space," a "set," etc.

1. PRELIMINARIES

1.1. Basic Braided Algebra

We choose, once and for all, a braided monoidal category $\mathfrak{C} = (\mathscr{C}, \otimes, \mathbf{1}, \Phi, \ell, r, \Psi)$ abiding by the following conditions:

(1.1) $\mathscr{C}$ is a subcategory of the category of all k-linear spaces with a field k.
(1.2) $\otimes$ is the standard tensor product of k-linear spaces.
(1.3) The unit object $\mathbf{1}$ is k, regarded as a k-linear space in the standard manner.
(1.4) The associativity constraint Φ, the left unit constraint ℓ and the right unit constraint r are the standard ones of k-linear spaces.
(1.5) There exists a finite set $\mathbf{\Pi}$ of mutually nonisomorphic objects of $\mathscr{C}$ including the unit object $\mathbf{1}$, say, $\mathbf{\Pi} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{k}\}$, such that
   (1.5.1) Every object $\mathbf{p}$ in $\mathbf{\Pi}$ is a one-dimensional k-linear space.
   (1.5.2) The set $\mathbf{\Pi}$ is closed under $\otimes$, i.e., for any objects $\mathbf{p}$, $\mathbf{q}$ in $\mathbf{\Pi}$, there exists an object $\mathbf{r}$ in $\mathbf{\Pi}$ such that $\mathbf{p} \otimes \mathbf{q}$ is isomorphic to $\mathbf{r}$ in the category $\mathscr{C}$ (we will constantly use $\mathbf{p}, \mathbf{q}, \mathbf{r}, \ldots$ with or without subscripts as variables over $\mathbf{\Pi}$).
   (1.5.3) Every direct sum of (possibly infinitely many) copies of objects in $\mathbf{\Pi}$ as well as all its associated canonical injections and projections belongs to $\mathscr{C}$, and any object in $\mathscr{C}$ is a direct sum of copies of objects in $\mathbf{\Pi}$.
(1.6) For any morphism $\alpha : U \to V$ in $\mathscr{C}$, if α happens to be an isomorphism of k-linear spaces, then $\alpha^{-1} : V \to U$ belongs to $\mathscr{C}$ so that α is an isomorphism in $\mathscr{C}$.

Note that we have not assumed Ψ to be symmetric. As is the custom in dealing with monoidal categories, we will often proceed as if the monoidal category $(\mathscr{C}, \otimes, \mathbf{1}, \Phi, \ell, r)$ were strict, which is justifiable by Theorem XI.5.3 of Kassel (1995). We will often write $\mathbf{p} + \mathbf{q}$ for $\mathbf{r}$ isomorphic to $\mathbf{p} \otimes \mathbf{q}$ in (1.5.2). Then it is easy to see that

Proposition 1.1. $\mathbf{\Pi}$ *is an abelian monoid with respect to the operation* $+$ *defined above.*

Proof: The associativity constraint $\Phi_{\mathbf{p},\mathbf{q},\mathbf{r}} : (\mathbf{p} \otimes \mathbf{q}) \otimes \mathbf{r} \to \mathbf{p} \otimes (\mathbf{q} \otimes \mathbf{r})$ guarantees that $\mathbf{\Pi}$ is a semigroup. The left unit constraint $\ell_{\mathbf{p}} : \mathbf{1} \otimes \mathbf{p} \to \mathbf{p}$ and the right unit constraint $r_{\mathbf{p}} : \mathbf{p} \otimes \mathbf{1} \to \mathbf{p}$ warrants that $\mathbf{\Pi}$ is not only a semigroup but also a monoid. The commutativity of the monoid $\mathbf{\Pi}$ follows from the braiding $\Psi_{\mathbf{p},\mathbf{q}} : \mathbf{p} \otimes \mathbf{q} \to \mathbf{q} \otimes \mathbf{p}$. $\square$

We choose an arbitrary nonzero element $x_{\mathbf{p}}$ of each one-dimensional $\mathscr{k}$-linear space $\mathbf{p}$ in $\mathbf{\Pi}$ once and for all. For $\mathbf{p}, \mathbf{q}$ in $\mathbf{\Pi}$ there exists a unique $\delta^{\mathbf{p},\mathbf{q}} \in \mathscr{k}$ such that

(1.7) $\Psi_{\mathbf{p},\mathbf{q}}(x_{\mathbf{p}} \otimes x_{\mathbf{q}}) = \delta^{\mathbf{p},\mathbf{q}}(x_{\mathbf{q}} \otimes x_{\mathbf{p}})$

It is easy to see that the numbers $\delta^{\mathbf{p},\mathbf{q}}$ do not depend on our particular choice of $\{x_{\mathbf{p}}\}_{\mathbf{p}\in\mathbf{\Pi}}$.

Proposition 1.2. *The numbers $\delta^{\mathbf{p},\mathbf{q}}$ satisfy the following identities:*

(1.8) $\delta^{\mathbf{p},\mathbf{q}+\mathbf{r}} = \delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{p},\mathbf{r}}$
(1.9) $\delta^{\mathbf{p}+\mathbf{q},\mathbf{r}} = \delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{r}}$
(1.10) $\delta^{\mathbf{p},\mathbf{1}} = \delta^{\mathbf{1},\mathbf{p}} = 1$

Proof: (1.8) and (1.9) follow from the so-called hexagon axiom, which claims that $\Psi_{\mathbf{p},\mathbf{q}\otimes\mathbf{r}} = (\mathrm{id}_{\mathbf{q}} \otimes \Psi_{\mathbf{p},\mathbf{r}}) \circ (\Psi_{\mathbf{p},\mathbf{q}} \otimes \mathrm{id}_{\mathbf{r}})$ and $\Psi_{\mathbf{p}\otimes\mathbf{q},\mathbf{r}} = (\Psi_{\mathbf{p},\mathbf{r}} \otimes \mathrm{id}_{\mathbf{q}}) \circ (\mathrm{id}_{\mathbf{p}} \otimes \Psi_{\mathbf{q},\mathbf{r}})$ up to associativity and unit constraints. Since $\delta^{\mathbf{p},\mathbf{1}} = \delta^{\mathbf{p},\mathbf{1}+\mathbf{1}} = \delta^{\mathbf{p},\mathbf{1}}\delta^{\mathbf{p},\mathbf{1}}$ by (1.8), it follows that $\delta^{\mathbf{p},\mathbf{1}} = 1$. Similary it follows from (1.9) that $\delta^{\mathbf{1},\mathbf{p}} = 1$. □

Since we do not assume Ψ to be symmetric, it does not follow that $\delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{q},\mathbf{p}} = 1$. We require that

(1.11) $\mathbf{\Pi}$ is not only an abelian monoid but even an abelian group, so that, if the braiding Ψ is symmetric, the pair $(\mathbf{\Pi}, \delta)$ is a signed group in terms of Marcinek (1991).

Given an object U in $\mathscr{C}$, the direct sum decomposition of U into objects in $\mathbf{\Pi}$ in (1.5.3) is not unique, but the $\mathbf{p}$-*component of* U defined as the direct sum of the images of all the canonical injections from $\mathbf{p}$ into U with respect to a particular decomposition of U, will soon turn out to be independent of our choice of a particular decomposition of U. Therefore we can safely write $U^{\mathbf{p}}$ for the $\mathbf{p}$-component of U.

Proposition 1.3. *Let Γ and Γ' be two direct sum decompositions of U in* (1.5.3). *Then for any* $\mathbf{p}$ *in* $\mathbf{\Pi}$, *the* $\mathbf{p}$-*components $U^{\mathbf{p}}_{\Gamma}$ and $U^{\mathbf{p}}_{\Gamma'}$ of U with respect to Γ and Γ' coincide.*

Proof: The proof uses a gimmick that is familiar in the proof of the well-known fact of algebra that although a direct sum decomposition of a semisimple module into simple ones is not unique, its homogeneous component affiliated to a particular simple module is well defined, for which the reader is referred, e.g., to Wisbauer (1991, Chapter 4). For any canonical injection ι of $\mathbf{p}$ into U in the decomposition Γ and any canonical projection π of U onto $\mathbf{q}$ in the decomposition Γ' with

$\mathbf{p} \neq \mathbf{q}$, $\pi \circ \iota = 0$, for otherwise $\mathbf{p}$ and $\mathbf{q}$ would be isomorphic in $\mathscr{C}$ by (1.7). This means that $U_{\Gamma}^{\mathbf{p}} \subset U_{\Gamma'}^{\mathbf{p}}$ for any $\mathbf{p}$ in $\mathbf{\Pi}$. By interchanging the roles of Γ and Γ' in the above discussion, we have that $U_{\Gamma'}^{\mathbf{p}} \subset U_{\Gamma}^{\mathbf{p}}$ for any $\mathbf{p}$ in $\mathbf{\Pi}$. Therefore the desired conclusion follows. □

Corollary 1.4. *$U = U^{\mathbf{1}} \oplus \cdots \oplus U^{\mathbf{k}}$, so that each $u \in U$ can be decomposed uniquely as $u = u_{\mathbf{1}} + \cdots + u_{\mathbf{k}}$ with $u_{\mathbf{p}} \in U^{\mathbf{p}}$ for any $\mathbf{p}$ in $\mathbf{\Pi}$.*

An element u of U which happens to consist in $U^{\mathbf{p}}$ for some $\mathbf{p}$ in $\mathbf{\Pi}$ is called *pure (of grade* $\mathbf{p}$), in which we will denote $\mathbf{p}$ by $|u|$.

The same gadget used in the proof of Proposition 1.3 establishes the following:

Proposition 1.5. *Any morphism $\alpha : U \to V$ in $\mathscr{C}$ preserves grading (i.e., $\alpha(U^{\mathbf{p}}) \subset V^{\mathbf{p}}$ for each $\mathbf{p}$ in $\mathbf{\Pi}$.*

We now enjoin that the class of morphisms in $\mathscr{C}$ be saturated with respect to this property in the following sense:

(1.12) For any objects U, V in $\mathscr{C}$, if a homomorphism $\alpha : U \to V$ of $\mathscr{k}$-linear spaces preserves grading (i.e., $\alpha(U^{\mathbf{p}}) \subset V^{\mathbf{p}}$ for any $\mathbf{p}$ in $\mathbf{\Pi}$), then α lies in $\mathscr{C}$.

The notion of an algebra in the braided monoidal category $\mathfrak{C}$, usually called a $\mathfrak{C}$-*algebra*, can be defined diagrammatically as in Kassel (1995, Section III.1). A $\mathfrak{C}$-algebra $\mathscr{A}$ with its product $\mu_{\mathscr{A}} : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ is said to be $\mathfrak{C}$-*commutative* if $\mu_{\mathscr{A}} \circ \Psi_{\mathscr{A},\mathscr{A}} = \mu_{\mathscr{A}}$. Given a $\mathfrak{C}$-algebra $\mathscr{A}$, the notions of a left $\mathscr{A}$-module and a right $\mathscr{A}$-module in $\mathfrak{C}$, usually called a *left* $\mathscr{A}$-$\mathfrak{C}$-*module* and a *right* $\mathscr{A}$-$\mathfrak{C}$-*module*, respectively, can be defined diagrammatically as in Majid (1995a, Section 1.6). If $\mathscr{A}$ happens to be $\mathfrak{C}$-commutative, a left $\mathscr{A}$-$\mathfrak{C}$-module $\mathscr{M}$ with its left action $\eta : \mathscr{A} \otimes \mathscr{M} \to \mathscr{M}$ can naturally be converted into a right $\mathscr{A}$-$\mathfrak{C}$-module with its right action $\eta \circ \Psi_{\mathscr{M},\mathscr{A}} : \mathscr{M} \otimes \mathscr{A} \to \mathscr{M}$, and vice versa, so that the distinction between "left" and "right" is not essential in the $\mathfrak{C}$-commutative case. In this case any left $\mathscr{A}$-$\mathfrak{C}$-module and, equivalently, any right $\mathscr{A}$-$\mathfrak{C}$-module are naturally $\mathscr{A}$-bimodules. A left (right, respectively) $\mathscr{A}$-$\mathfrak{C}$-module $\mathscr{M}$ is said to be $\mathfrak{C}$-*finite-dimensional* if there exists a finite-dimensional $\mathscr{k}$-linear space V in $\mathscr{C}$ such that $\mathscr{A} \otimes V$, ($V \otimes \mathscr{A}$, respectively) is isomorphic to $\mathscr{M}$ as left (right, respectively) $\mathscr{A}$-$\mathfrak{C}$-modules. The notions of a left $\mathscr{A}$-module algebra and a right $\mathscr{A}$-module algebra in $\mathfrak{C}$, usually called a *left* $\mathscr{A}$-$\mathfrak{C}$-*algebra* and a *right* $\mathscr{A}$-$\mathfrak{C}$-*algebra*, respectively, can also be defined diagrammatically as in Majid (1995a, Section 1.6). An ideal of a $\mathfrak{C}$-algebra $\mathscr{A}$ is said to be a $\mathfrak{C}$-*ideal* if it belongs to $\mathscr{C}$. Other standard notions such as that of a *homomorphism of* $\mathfrak{C}$-*algebras* that can easily be formulated diagrammatically will be used freely. Given a $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra $\mathscr{A}$ and an $\mathscr{A}$-$\mathfrak{C}$-algebra $\mathscr{B}$, $\mathrm{Spec}_{\mathscr{A}}\,\mathscr{B}$ denotes the totality of homomorphisms of $\mathscr{A}$-$\mathfrak{C}$-algebra from $\mathscr{B}$ into $\mathscr{A}$.

Now we choose, once and for all, a $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra $\mathbb{R}$ intended to play a role of real numbers in our braided mathematics. So we must enjoin the following axiom on $\mathbb{R}$:

(1.13) $\mathbb{R}$ is a $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra.

Another important axiom on $\mathbb{R}$ will be presented in the next subsection. Given a set Z, the totality of functions from Z to $\mathbb{R}$ is an $\mathbb{R}$-$\mathfrak{C}$-algebra with componentwise operations whose **p**-component can naturally be identified with the totality of functions from Z to $\mathbb{R}^{\mathbf{p}}$.

Given right $\mathbb{R}$-$\mathfrak{C}$-modules $\mathscr{M}$ and $\mathscr{N}$, the totality $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$ of $\mathbb{R}$-homomorphisms from $\mathscr{M}$ to $\mathscr{N}$ is a left $\mathbb{R}$-module in the sense that for any $a \in \mathbb{R}$, any $u \in \mathscr{M}$ and any $f \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$,

(1.14) $(af)(u) = af(u)$.

It is not difficult to see that the **p**-component $\mathrm{Hom}^{\mathbf{p}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$ of $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$ is the totality of $f \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$ such that $|f(u)| = |u| + \mathbf{p}$ for any $u \in \mathscr{M}$. We define $\mathrm{Hom}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$ to be $\mathrm{Hom}^{\mathbf{1}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N}) \oplus \cdots \oplus \mathrm{Hom}^{\mathbf{K}}_{\mathbb{R}}(\mathscr{M}, \mathscr{N})$, which is an $\mathbb{R}$-$\mathfrak{C}$-module.

Given a finite sequence $\mathbf{p}_1, \ldots, \mathbf{p}_n$ in $\mathbf{\Pi}$, we can form the tensor $\mathfrak{C}$-algebra $T(\mathbf{p}_1 \oplus \cdots \oplus \mathbf{p}_n)$ of the $\mathscr{k}$-linear space $\mathbf{p}_1 \oplus \cdots \oplus \mathbf{p}_n$. The quotient $\mathfrak{C}$-algebra of $T(\mathbf{p}_1 \oplus \cdots \oplus \mathbf{p}_n)$ with respect to the $\mathfrak{C}$-ideal generated by $\{x_{\mathbf{p}_j} x_{\mathbf{p}_i} - \delta^{\mathbf{p}_j, \mathbf{p}_i} x_{\mathbf{p}_i} x_{\mathbf{p}_j} | 1 \leq i, \ j \leq n\}$ is a $\mathfrak{C}$-algebra called the *polynomial* $\mathfrak{C}$*-algebra of variables* $x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}$ and is denoted by $\mathscr{k}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$. The $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathbb{R} \underline{\otimes}$[2] $\mathscr{k}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ is called the *polynomial* $\mathfrak{C}$*-algebra of variables* $x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}$ *over* $\mathbb{R}$ or the *polynomial* $\mathbb{R}$-$\mathfrak{C}$*-algebra of variables* $x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}$ and is denoted by $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$. The $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathbb{R}[x_{\mathbf{p}_1}, \ldots x_{\mathbf{p}_n}]$ is characterized by the following universality property:

Proposition 1.6. *The* $\mathbb{R}$-$\mathfrak{C}$*-algebra* $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ *is* $\mathfrak{C}$*-commutative. For any* $\mathfrak{C}$*-commutative* $\mathbb{R}$-$\mathfrak{C}$*-algebra* $\mathscr{A}$ *and any morphisms* $\alpha_i : \mathbf{p}_i \to \mathscr{A}$ *in* $\mathscr{C}$ $(1 \leq i \leq n)$*, there exists a unique homomorphism* α *of* $\mathbb{R}$-$\mathfrak{C}$*-algebras from* $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ *to* $\mathscr{A}$ *whose restriction to* $\mathbf{p}_i$ *is* α_i *(*$1 \leq i \leq n$*).*

1.2. Weil $\mathfrak{C}$-Algebras and $\mathfrak{C}$-Microlinearity

A *Weil* $\mathfrak{C}$*-algebra* is a $\mathfrak{C}$-commutative $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathfrak{W}$ which, regarded as an $\mathbb{R}$-module, is to be written as $\mathfrak{W} = \mathbb{R} \oplus \mathfrak{m}$ with the first component being the $\mathbb{R}$-$\mathfrak{C}$-algebra structure and the second being a finite-dimensional nilpotent $\mathfrak{C}$-ideal (called the $\mathfrak{C}$*-ideal of augmentation*). By way of example, the quotient $\mathfrak{C}$-algebra of the polynomial $\mathfrak{C}$-algebra $\mathbb{R}[X_1, \ldots, X_n]$ with respect to the $\mathfrak{C}$-ideal generated by $\{X_i X_j | 1 \leq i \leq n\}$ is a Weil $\mathfrak{C}$-algebra and is denoted by $\mathfrak{W}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ with

[2] $\underline{\otimes}$ denotes the braided tensor product.

$\mathbf{p}_i = |X_i|$ $(1 \leq i \leq n)$. Given Weil $\mathfrak{C}$-algebras $\mathfrak{W}_1$ and $\mathfrak{W}_2$ with their $\mathfrak{C}$-ideals of augmentation $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively, a homomorphism of $\mathbb{R}$-$\mathfrak{C}$-algebras $\varphi : \mathfrak{W}_1 \to \mathfrak{W}_2$ is said to be a *homomorphism of Weil* $\mathfrak{C}$*-algebras* if it preserves their $\mathfrak{C}$-ideals of augmentation, i.e., if $\varphi(\mathfrak{m}_1) \subset \mathfrak{m}_2$. A finite limit diagram of $\mathbb{R}$-$\mathfrak{C}$-algebras is said to be a *good finite limit diagram of Weil* $\mathfrak{C}$*-algebras* if every object occurring in the diagram is a Weil $\mathfrak{C}$-algebra and every morphism occurring in the diagram is a homomorphism of Weil $\mathfrak{C}$-algebras. The diagram obtained from a good finite limit diagram of Weil $\mathfrak{C}$-algebras by taking $\mathrm{Spec}_{\mathbb{R}}$ is called a *quasi-colimit diagram of* $\mathfrak{C}$*-small objects.*

The Braided version of the general Kock axiom, called the *general* $\mathfrak{C}$*-Kock axiom*, is as follows:

(1.15) For any Weil $\mathfrak{C}$-algebra $\mathfrak{W}$, the canonical homomorphism $\mathfrak{W} \to \mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(\mathfrak{W})}$ of $\mathbb{R}$-$\mathfrak{C}$-algebras is an isomorphism.

Spaces of the form $\mathrm{Spec}_{\mathbb{R}}$ $(\mathfrak{W})$ for some Weil $\mathfrak{C}$-algebra $\mathfrak{W}$ are called $\mathfrak{C}$-*infinitesimal spaces* or $\mathfrak{C}$*-small objects*. The $\mathfrak{C}$-infinitesimal space corresponding to Weil $\mathfrak{C}$-algebra $\mathfrak{W}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is denoted by $D(\mathbf{p}_1, \ldots, \mathbf{p}_n)$. By Proposition 1.6, $D(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is to be identified with $\{(d_1, \ldots, d_n) \mid d_i \in \mathbb{R}^{\mathbf{p}_i}\ (1 \leq i \leq n),\ d_i d_j = 0\ (1 \leq i,\ j \leq n)\}$. We will often denote $D(\mathbf{p})$ by $D^{\mathbf{p}}$.

The $\mathfrak{C}$-infinitesimal space $D(\mathbf{1}, \ldots, \mathbf{k})$ plays a very important role in our discussion of tangency. First we note that $D(\mathbf{1}, \ldots, \mathbf{k})$ can be identified with the subset of $\mathbb{R}$ consisting of all $\mathbb{d} \in \mathbb{R}$ such that $\mathbb{d}_{\mathbf{p}}\mathbb{d}_{\mathbf{q}} = 0$ for any $\mathbf{p},\ \mathbf{q} \in \mathbf{\Pi}$. Under this identification $(d_1, \ldots, d_k) \in D$ $(\mathbf{1}, \ldots \mathbf{k})$ corresponds to $d_1 + \cdots + d_k \in \mathbb{R}$. What concerns us most about $D(\mathbf{1}, \ldots, \mathbf{k})$ is that the space $D(\mathbf{1}, \ldots, \mathbf{k})$, regarded as a subset of $\mathbb{R}$, is closed under the left and right actions of $\mathbb{R}$ on itself.

Just as the general Kock axiom paved to the introduction of microlinear spaces, its braided version invokes the notion of a $\mathfrak{C}$*-microlinear space*, which is by definition a space M abiding by the following condition:

(1.16) For any good finite limit diagram of Weil $\mathfrak{C}$-algebras with its limit W, the diagram obtained by taking $\mathrm{Spec}_{\mathbb{R}}$ and then exponentiating over M is a limit diagram with its limit $M^{\mathrm{Spec}_{\mathbb{R}} W}$.

The following proposition guarantees that we have a plenty of $\mathfrak{C}$-microlinear spaces.

Proposition 1.7.

(1) $\mathbb{R}^{\mathbf{p}}$ *is a* $\mathfrak{C}$*-microlinear space for any* $\mathbf{p} \in \mathbf{\Pi}$.
(2) *The class of* $\mathfrak{C}$*-microlinear spaces is closed under limits and exponentiation by an arbitrary space.*

The above braided version of the general Kock axiom surely subsumes the following braided version of the Kock–Lawvere axiom to be called the

$\mathfrak{C}$-*Kock–Lawvere* axiom:

(1.17) For any function $f : D^{\mathbf{p}} \to \mathbb{R}$, there exists unique $b \in \mathbb{R}$ such that $f(d) = f(0) + bd$ for any $d \in D^{\mathbf{p}}$.

The axiom (1.17) is equivqlent to the following axiom:

(1.18) For any function $f : D^{\mathbf{p}} \to \mathbb{R}$, there exists unique $b' \in \mathbb{R}$ such that $f(d) = f(0) + db'$ for any $d \in D^{\mathbf{p}}$.

We conclude this subsection by a definition. An $\mathbb{R}$-$\mathfrak{C}$-module $\mathscr{M}$ is said to be $\mathfrak{C}$-*Euclidean* if it abides by the following equivalent of (1.17):

(1.19) For any function $f : D^{\mathbf{p}} \to \mathscr{M}$, there exists unique $x \in \mathscr{M}$ such that $f(d) = f(0) + xd$ for any $d \in D^{\mathbf{p}}$.

1.3. $\mathfrak{C}$-Microcubes

A $\mathfrak{C}$-microlinear space M shall be chosen arbitrarily ones and for all. Given $(\mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathbf{\Pi}^n$, a *pure* n-$\mathfrak{C}$-*microcube of type* $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ *on* M is a function from $D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$ to M. We denote by $\mathbf{T}^{\mathbf{p}_1,\dots,\mathbf{p}_n} M$ the totality of pure n-$\mathfrak{C}$-microcubes of type $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ on M. We denote by $\mathbf{T}^n M$ the set-theoretic union of $\mathbf{T}^{\mathbf{p}_1,\dots,\mathbf{p}_n} M$ for all $(\mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathbf{\Pi}^n$. In particular, $\mathbf{T}^1 M$ is usually denoted by **T**M, and their elements are called *pure* $\mathfrak{C}$-*vectors tangent to* M. Given $\gamma \in \mathbf{T}^{\mathbf{p}_1,\dots,\mathbf{p}_n} M$ and $e \in D^{\mathbf{p}_i}$, γ_e^i denotes the mapping $(d_i, \dots, d_{n-1}) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \cdots \times D^{\mathbf{p}_n} \mid\to \gamma(d_1, \dots, d_{i-1}, e, d_{i+1}, \dots, d_{n-1})$, which is surely a pure $(n-1)$-$\mathfrak{C}$-microcube of type $(\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \dots, \mathbf{p}_n)$.

An n-$\mathfrak{C}$-*microcube on* M is a mapping from $D(\mathbf{1}, \dots, \mathbf{k})^n$ to M. We denote by $\mathfrak{T}^n M$ the totality of n-$\mathfrak{C}$-microcubes on M. In particular, $\mathfrak{T}^1 M$ is usually denoted by $\mathfrak{T}M$ and their elements are called $\mathfrak{C}$-*vectors tangent to* M. Given $x \in M$, we denote the sets $\{t \in \mathbf{T}^{\mathbf{p}} M \mid t(0) = x\}$ and $\{\mathbb{t} \in \mathfrak{T}\ M \mid \mathbb{t}\ (0) = x\}$ by $\mathbf{T}_x^{\mathbf{p}} M$ and $\mathfrak{T}_x M$, respectively. We have shown (Nishimura, 1998b, Section 4) that $\mathfrak{T}_x M$ is an $\mathbb{R}$-$\mathfrak{C}$-module and that its $\mathbf{p}$-component can naturally be identified with $\mathbf{T}_x^{\mathbf{p}} M$. We have noted there also that the $\mathbb{R}$-$\mathfrak{C}$-module $\mathfrak{T}_x M$ is $\mathfrak{C}$-Euclidean.

Given $(\mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathbf{\Pi}^n$, the canonical injection of $D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$ into $D(\mathbf{1}, \dots, \mathbf{k})^n$ and the canonical projection of $D(\mathbf{1}, \dots, \mathbf{k})^n$ onto $D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$ are denoted by $\iota_{\mathbf{p}_1,\dots,\mathbf{p}_n}$ and $\pi_{\mathbf{p}_1,\dots,\mathbf{p}_n}$, respectively. The totality of $\overline{\gamma} \in \mathfrak{T}^n M$ with $\overline{\gamma} \circ \iota_{\mathbf{p}_1,\dots,\mathbf{p}_n} \circ \pi_{\mathbf{p}_1,\dots,\mathbf{p}_n} = \overline{\gamma}$ can and shall hereafter be identified with $\mathbf{T}^{\mathbf{p}_1,\dots,\mathbf{p}_n} M$.

1.4. $\mathfrak{C}$-Vector Bundles

A mapping $\zeta : E \to M$ of $\mathfrak{C}$-microlinear spaces is called a $\mathfrak{C}$-*vector bundle* providing that $E_x = \zeta^{-1}(x)$ is a $\mathfrak{C}$-Euclidean $\mathbb{R}$-$\mathfrak{C}$-module for any $x \in M$. We call M the *base space* of ζ and E_x the *fiber over* x. The totality of mappings $\lambda : M \to E$

with $\zeta \circ \lambda = \mathrm{id}_M$ ($\mathrm{id}_{\underline{M}}$ denotes the identity transformation of M) is denoted by Sec ζ. The totality of $\mathbb{t} \in \mathfrak{T}E$ with $\zeta \circ \mathbb{t} = \mathbb{0}$ (the zero $\mathfrak{C}$-vector to M at $\zeta \circ \mathbb{t}(0)$) is to be put down as a $\mathfrak{C}$-vector bundle over E and is to be denoted by $V(E)$.

The *tangent bundle* $\tau_M : M^{D(\mathbf{1},\ldots,\mathbf{k})} \to M$ is a $\mathfrak{C}$-vector bundle, where τ_M assigns, to each $\mathbb{t} \in M^{D(1,\ldots,\mathbf{k})}$, $\mathbb{t}(0) \in M$. If $\mathscr{N}$ is a $\mathfrak{C}$-Euclidean $\mathbb{R}$-$\mathfrak{C}$-module, which is a $\mathfrak{C}$-microlinear space, then the trivial bundle $M \times \mathscr{N} \to M$ is a $\mathfrak{C}$-vector bundle.

Various algebraic constructions in linear $\mathfrak{C}$-algebra can be carried over to $\mathfrak{C}$-vector bundles. If $\zeta : E \to M$ and $\eta : F \to M$ are $\mathfrak{C}$-vector bundles over the same base space M, then their Whitney sum $\zeta \oplus \eta$ and the natural projection $\pi_{\mathscr{L}(\zeta,\eta)} : \mathscr{L}(\zeta, \eta) \to M$ are $\mathfrak{C}$-vector bundles, where $\mathscr{L}(\zeta, \eta)$ denotes the set-theoretic union of $\mathrm{Hom}(\zeta_x, \eta_x)$ for all $x \in M$.

2. BRAIDED EXTERIOR DIFFERENTIAL CALCULUS

Given $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n} M$ and $a \in \mathbb{R}^{\mathbf{q}}$, pure n-$\mathfrak{C}$-microcubes $\gamma \dot{_i} a$ and $a \dot{_i} \gamma$ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_i - \mathbf{q}, \ldots, \mathbf{p}_n)$ on M ($1 \leq i \leq n$) are defined to be

(2.1) $(\gamma \dot{_i} a)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, a d_i, \ldots, d_n)$

(2.2) $(a \dot{_i} \gamma)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_i a, \ldots, d_n)$

for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_i - \mathbf{q}} \times \cdots \times D^{\mathbf{p}_n}$.

Let $\mathfrak{Sym}_n$ be the symmetric group of the set $\{1, \ldots, n\}$. Given $\gamma \in \mathbf{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n} M$ and $\sigma \in \mathfrak{Sym}_n$, a pure n-$\mathfrak{C}$-microcube $\Sigma_\sigma(\gamma)$ of type $(\mathbf{p}_{\sigma^{-1}(1)}, \ldots, \mathbf{p}_{\sigma^{-1}(n)})$ on M is defined as follows:

(2.3) $\Sigma_\sigma(\gamma)(d_1, \ldots, d_n) = \gamma(d_{\sigma(1)}, \ldots, d_{\sigma(n)})$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_{\sigma^{-1}(1)}} \times \cdots \times D^{\mathbf{p}_{\sigma^{-1}(n)}}$.

A *(differential) n-$\mathfrak{C}$-preform on M* is a mapping θ from $\mathbf{T}^n M$ to $\mathbb{R}$ abiding by the following conditions:

(2.4) $\theta(\gamma \dot{_i} a) = \theta(a \dot{_{i+1}} \gamma)$ $(1 \leq i \leq n-1)$ while $\theta(\gamma \dot{_n} a) = \theta(\gamma)\, a$ for any $a \in \mathbb{R}^1$ and any $\gamma \in \mathbf{T}^n M$;

(2.5) If γ is a pure n-$\mathfrak{C}$-microsquare of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, then $\theta(\gamma) = -\delta^{\mathbf{p}_i,\mathbf{p}_{i+1}} \theta(\Sigma_{(i,i+1)}(\gamma))$ $(1 \leq i \leq n-1)$, where $(i, i+1)$ is the transposition of i and $i+1$.

A differential n-$\mathfrak{C}$-preform θ on M is called *$\mathfrak{C}$-braided* if it abides by the following condition:

(2.6) $\theta(\gamma \dot{_i} a) = \theta(a \dot{_{i+1}} \gamma)$ $(1 \leq i \leq n-1)$ while $\theta(\gamma \dot{_n} a) = \theta(\gamma)\, a$ for any $\mathbf{p} \in \mathbf{\Pi}$, $a \in \mathbb{R}^{\mathbf{p}}$ and any $\gamma \in \mathbf{T}^n M$.

We denote by $\underline{\Xi}_n(M)$ and $\underset{\sim}{\Xi}_n(M)$ the totality of differential n-$\mathfrak{C}$-preforms on M and that of $\mathfrak{C}$-braided differential n-$\mathfrak{C}$-preforms on M, respectively.

Given $\overline{\gamma} \in \mathfrak{T}^n M$ and $a \in \mathbb{R}$, n-$\mathfrak{C}$-microcubes $\gamma \underset{i}{\cdot} a$ and $a \underset{i}{\cdot} \overline{\gamma}$ on M $(1 \leq i \leq n)$ are defined as in (2.1) and (2.2), respectively. Given $\overline{\gamma} \in \mathfrak{T}^n M$ and $\sigma \in \mathfrak{Sym}_n$, an n-$\mathfrak{C}$-microcube $\underline{\Sigma}_\sigma(\overline{\gamma})$ on M is defined as in (2.3). A (*differential*) *n-$\mathfrak{C}$-form on* M is a mapping $\overline{\theta}$ from $\mathfrak{T}^n M$ to $\mathbb{R}$ subject to the following conditions:

(2.7) $\overline{\theta}(\overline{\gamma} \underset{i}{\cdot} a) = \overline{\theta}(a \underset{i+1}{\cdot} \overline{\gamma})$ $(1 \leq i \leq n-1)$ while $\overline{\theta}(\overline{\gamma} \underset{n}{\cdot} a) = \overline{\theta}(\overline{\gamma})\, a$ for any $a \in \mathbb{R}$ and any $\overline{\gamma} \in \mathfrak{T}^n M$;

(2.8) If $\overline{\gamma}$ is a pure n-$\mathfrak{C}$-micosquare of type $(p_1, \ldots, p_n)$ on M, then $\overline{\theta}(\overline{\gamma}) = -\delta^{\mathbf{p}_i, \mathbf{p}_{i+1}} \overline{\theta}(\underline{\Sigma}_{(i,i+1)}(\overline{\gamma}))$ $(1 \leq i \leq n-1)$.

We denote by $\tilde{\Xi}_n(M)$ the totality of differential n-$\mathfrak{C}$-forms on M.

Proposition 2.1. *There is a natural bijective correspondence between* $\underset{\sim}{\Xi}_n(M)$ *and* $\tilde{\Xi}_n(M)$.

Proof: By the same token as in Nishimura (1999, Proposition 1.2). □

Therefore we can loosely and will often identify $\mathfrak{C}$-braided differential n-$\mathfrak{C}$-preforms and differential n-$\mathfrak{C}$-forms, so that we will loosely denote $\underset{\sim}{\Xi}_n(M)$ and $\tilde{\Xi}_n(M)$ by the same symbol $\Xi_n(M)$.

A *marked pure n-$\mathfrak{C}$-microcube of type* $(\mathbf{p}_1, \ldots, \mathbf{P}_n)$ on M is a pair $(\gamma, \underline{e})$ of a pure n-$\mathfrak{C}$-microcube γ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M and $\underline{e} = (e_1, \ldots, e_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$. We denote by $\tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_n} M$ the totality of marked pure n-$\mathfrak{C}$-microcubes of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M. We denote by $\tilde{\mathbf{T}}^n M$ the set-theoretic union of $\tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_n} M$ for all $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbf{\Pi}^n$.

Given $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbf{\Pi}^n$, $(\gamma, \underline{e}) \in \tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_n} M$ and $\theta \in \underset{\sim}{\Xi}_n(M)$ with $\underline{e} = (e_1, \ldots, e_n)$, $\varphi_\theta(\gamma, \underline{e}) \in \mathbb{R}$ is defined as follows:

(2.9) $\varphi_\theta(\gamma, \underline{e}) = \theta(\gamma)\, e_1, \ldots, e_n$

Given $\underline{e} = (e_1, \ldots, e_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$ and $a \in \mathbb{R}^{\mathbf{q}}$, elements $a \underset{i}{\cdot} \underline{e}$ and and $\underline{e} \underset{i}{\cdot} a$ of $D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_i + \mathbf{q}} \times \cdots \times D^{\mathbf{p}_n} (1 \leq i \leq n)$ are defined to be

(2.10) $a \underset{i}{\cdot} \underline{e} = (e_1, \ldots, a e_i, \ldots, e_n)$

(2.11) $\underline{e} \underset{i}{\cdot} a = (e_1, \ldots, e_i a, \ldots, e_n)$

Given $\underline{e} = (e_1, \ldots, e_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$ and $\sigma \in \mathfrak{Sym}_n$, $\Sigma_\sigma(e) \in D^{\mathbf{p}_{\sigma^{-1}(1)}} \times \cdots \times D^{\mathbf{p}_{\sigma^{-1}(n)}}$ is defined to be

(2.12) $\sum_\sigma(e) = (e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(n)})$

Given $\theta \in \underset{\sim}{\Xi}_n(M)$, it is easy to see that the function $\varphi_\theta \cdot \tilde{\mathbf{T}}^n(M) \to \mathbb{R}$ satisfies the following properties with $(\gamma, \underline{e}) \in \tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_n} M$ and $a \in \mathbb{R}^1$.

(2.13) $\varphi_\theta(\gamma \underset{i}{\cdot} a, \underline{e}) = \varphi_\theta(a \underset{i}{\cdot} \gamma, \underline{e}) = a\varphi_\theta(\gamma, \underline{e}) \quad (1 \leq i \leq n)$

(2.14) $\varphi_\theta(\gamma, \underline{e}_i a) = \varphi_\theta(\gamma, a_i \underline{e}) = a\varphi_\theta(\gamma, \underline{e}) \quad (1 \leq i \leq n)$
(2.15) $\varphi_\theta(\gamma, \underline{e}) = -\delta^{\mathbf{p}_i,\mathbf{p}_{i+1}}\delta^{\mathbf{q}_i,\mathbf{q}_{i+1}}\varphi_\theta(\Sigma_{(i,i+1)}(\gamma), \Sigma_{(i,i+1)}(\underline{e})) \quad (1 \leq i \leq n-1)$

Now we have the following converse.

Proposition 2.2. *If a function φ from $\tilde{\mathbf{T}}^n M$ to $\mathbb{R}$ abides by conditions* (2.13)–(2.15), *then there exists unique $\theta \in \underline{\Xi}_n(M)$ such that $\varphi = \varphi_\theta$.*

Proof: By the same token as in Lavendhomme (1996, Section 4.2, Proposition 2). □

Given $\theta \in \underline{\Xi}_n(M)$, we define ψ_θ to be the function from $\tilde{\mathbf{T}}^{n+1}(M)$ to $\mathbb{R}$ such that for any $(\gamma, \underline{e}) \in \tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}} M$ with $\underline{e} = (e_1, \ldots, e_{n+1})$,

(2.16) $\psi_\theta(\gamma, \underline{e}) = \Sigma_{i=1}^{n+1}(-1)^i \alpha_i (\theta(\gamma_0^i) - \theta(\gamma_{e_i}^i))e_1 \ldots \hat{e}_i \ldots e_{n+1}$

where $\alpha_i = (\Pi_{j=i+1}^{n+1} \delta^{\mathbf{p}_i,\mathbf{p}_j})(\Pi_{k=1}^{i-1} \delta^{\mathbf{p}_k,\mathbf{p}_i})$.

Proposition 2.3. *The above function $\psi_\theta : \tilde{\mathbf{T}}^{n+1} M \to \mathbb{R}$ satisfies conditions* (2.13)–(2.15).

Proof: By the same token as in Lavendhomme (1996, Section 4.2, Proposition 3). □

We denote by $\mathbf{d}\theta$ the element of $\underline{\Xi}_{n+1}(M)$ such that $\varphi_{d\theta} = \psi_\theta$. Its existence and uniqueness is guaranteed by Propositions 2.2 and 2.3. It is called the *exterior* $\mathfrak{C}$*-derivative* of θ. Now we have a family $\{\mathbf{d} : \underline{\Xi}_n(M) \to \underline{\Xi}_{n+1}(M)\}_{n \in \mathbb{N}}$ of mappings, for which we have the following:

Proposition 2.4. $\mathbf{d} \circ \mathbf{d} = 0$.

Proof: By the same token as in Lavendhomme (1996, Section 4.2, Proposition 1). □

Proposition 2.5. *If a differential n-$\mathfrak{C}$-preform θ on M is $\mathfrak{C}$-braided, then so is* $\mathbf{d}\theta$.

Proof: By the same token as in Nishimura (1999, Proposition 2.5 and Lemmas 2.6 and 2.7; 2000a, Lemmas 2.3 and 2.4). □

Therefore the family $\{\mathbf{d} : \underline{\Xi}_n(M) \to \underline{\Xi}_{n+1}(M)\}_{n \in \mathbb{N}}$ of mappings naturally gives rise to a family $\{\mathbf{d} : \Xi_n(M) \to \Xi_{n+1}(M)\}_{n \in \mathbb{N}}$ of mappings.

If $\varphi : M \to N$ a map of $\mathfrak{C}$-microlinear space and $\eta : F \to N$ is a $\mathfrak{C}$-vector bundle, then the notion of a ($\mathfrak{C}$-braided) differential n-$\mathfrak{C}$-preform on M and that of differential n-$\mathfrak{C}$-form on M discussed earlier can be generalized easily to that of a (*$\mathfrak{C}$-braided*) *differential* n-*$\mathfrak{C}$-preform on M with values in η relative to φ* and that of a *differential n-form on M with values in η relative to φ*, as in Lavendhomme (1996, Section 5.3.1). We denote by $\underset{\sim}{\Xi}^n(M \overset{\varphi}{\to} N; \eta), \underset{\approx}{\Xi}^n(M \overset{\varphi}{\to} N; \eta)$, and $\widetilde{\Xi}^n(M \overset{\varphi}{\to} N; \eta)$ the totality of differential n-$\mathfrak{C}$-prefroms on M with values in η relative to φ, that of $\mathfrak{C}$-braided differential n-$\mathfrak{C}$-preforms on M with values in η relative to φ, and that of differential n-$\mathfrak{C}$-forms on M with values in η relative to φ, respectively. A direct generalization of Proposition 2.1 enables us to identity $\underset{\approx}{\Xi}^n(M \overset{\varphi}{\to} N; \eta)$ and $\widetilde{\Xi}^n(M \overset{\varphi}{\to} N; \eta)$, which we will often denote by $\Xi^n(M \overset{\varphi}{\to} N; \eta)$. If $N = M$ and φ is the identity map id_M of M, then $\Xi^n(M \overset{\varphi}{\to} N; \eta)$ is denoted also by $\Xi^n(M; \eta)$. If η is furthermore a trivial bundle $M \times \mathbb{R} \to M$, then $\Xi^n(M; \eta)$ degenerates into $\Xi^n(M)$.

3. BRAIDED CONNECTIONS

Let $\zeta : E \to M$ be a $\mathfrak{C}$-vector bundle. A *$\mathfrak{C}$-connection* on ζ is a mapping $\nabla : M^{D(\mathbf{1},\ldots,\mathbf{k})} \times_M E \to E^{D(\mathbf{1},\ldots,\mathbf{k})}$ such that for any $(\mathbb{t}, v) \in M^{D(\mathbf{1},\ldots,\mathbf{k})} \times_M E$, any $a \in \mathbb{R}$ and any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$ we have that

(3.1) $\nabla(\mathbb{t}, v)(0) = v$

(3.2) $\nabla(\mathbb{t}\, a, v)(\mathbb{d}) = \nabla(\mathbb{t}, v)(a\mathbb{d})$

(3.3) $\nabla(\mathbb{t}, va)(\mathbb{d}) = (\nabla(\mathbb{t}, v)(\mathbb{d}))a$

(3.4) The mapping $u \in E_{\mathbb{t}(0)} \mid\to \nabla(\mathbb{t}, u)(\mathbb{d}) \in E_{\mathbb{t}(\mathbb{d})}$, denoted by $\mathrm{p}^{\nabla}_{(\mathbb{t},\mathbb{d})}$ or $\mathrm{p}_{(\mathbb{t},\mathbb{d})}$, is bijective and preservs grades (i.e., $\mathrm{p}_{(\mathbb{t},\mathbb{d})}(E^{\mathbf{p}}_{\mathbb{t}(0)}) = E^{\mathbf{p}}_{\mathbb{t}(\mathbb{d})}$ for any $\mathbf{p} \in \mathbf{\Pi}$). Its inverse is denoted by $q^{\nabla}_{(\mathbb{t},\mathbb{d})} = \mathrm{q}_{(\mathbb{t},\mathbb{d})} : E_{\mathbb{t},(\mathbb{d})} \to E_{\mathbb{t}(0)}$. We call $\mathrm{p}_{(\mathbb{t},\mathbb{d})}$ the *parallel transport* from $\mathbb{t}(0)$ to $\mathbb{t}(\mathbb{d})$ along $\mathbb{t}$, while $\mathrm{q}_{(\mathbb{t},\mathbb{d})}$ is called the *parallel transport* from $\mathbb{t}(\mathbb{d})$ to $\mathbb{t}(0)$ along $\mathbb{t}$.

If the $\mathfrak{C}$-vector bundle $\zeta : E \to M$ is a trivial bundle $M \times \mathscr{N} \to M$, and if $\nabla(\mathbb{t}, (\mathbb{t}(0), x))(\mathbb{d}) = (\mathbb{t}(\mathbb{d}), x)$ for any $\mathbb{t} \in M^{D(\mathbf{1},\ldots,\mathbf{k})}$ any $x \in \mathscr{N}$ and any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$, then the $\mathfrak{C}$-connection ∇ is called *trivial*.

Given $\mathbb{t} \in E^{D(\mathbf{1},\ldots,\mathbf{k})}$, we define $\overline{\omega}(\overline{\mathbb{t}} \in E^{D(1,\ldots,\mathrm{k})}$ to be

(3.5) $\overline{\omega}(\overline{\mathbb{t}} = \overline{\mathbb{t}} - \nabla(\zeta \circ \overline{\mathbb{t}}, \overline{\mathbb{t}}(0))$

Since $\overline{\omega}(\overline{\mathbb{t}}) \in V(E)$, there exist unique $\omega_{\mathbf{p}}(\overline{\mathbb{t}}) \in E_{\zeta \circ \overline{\mathbb{t}}(0)} (\mathbf{p} \in \mathbf{\Pi})$ such that

(3.6) $\overline{\omega}(\overline{\mathbb{t}})(\mathbb{d}) = \overline{\mathbb{t}}(0) + \Sigma_{\mathbf{p} \in \mathbf{\Pi}} \omega_{\mathbf{p}}(\overline{\mathbb{t}})\mathbb{d}_{\mathbf{p}}$

for any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$. We define $\omega(\overline{\mathbb{t}})$ to be $\Sigma_{\mathbf{p} \in \mathbf{\Pi}} \omega_{\mathbf{p}}(\overline{\mathbb{t}})$.

Proposition 3.1. *Given $v \in E$ and $x \in M$ with $x = \zeta(V)$, the mapping $\overline{\mathbb{t}} \in (E^{D(\mathbf{1},\ldots,\mathbf{k})})_V \mid \to \omega(\overline{\mathbb{t}}) \in E$ satisfies* (2.6), *so that ω is a differential* 1-$\mathfrak{C}$*-form on E with values in ζ relative to ζ.*

Proof: By (3.2) $\bar{\omega}$ satisfies (2.6), so that for any $a \in \mathbb{R}$ and any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$,

$$
\begin{aligned}
(3.7)\quad & \bar{\omega}(\overline{\mathbb{t}a})(\mathbb{d}) \\
&= \bar{\omega}(\overline{\mathbb{t}})(a\mathbb{d}) \\
&= \bar{\omega}(\overline{\mathbb{t}})(\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\Sigma_{\mathbf{q}+\mathbf{r}=\mathbf{p}}a_{\mathbf{q}}\mathbb{d}_{\mathbf{r}}) \\
&= \overline{\mathbb{t}}(0) + \Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})(\Sigma_{\mathbf{q}+\mathbf{r}=\mathbf{p}}a_{\mathbf{q}}\mathbb{d}_{\mathbf{r}}) \\
&= \overline{\mathbb{t}}(0) + \Sigma_{\mathbf{r}\in\mathbf{\Pi}}(\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})a_{\mathbf{p}-\mathbf{r}})\mathbb{d}_{\mathbf{r}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(3.8)\quad \omega(\overline{\mathbb{t}a}) &= \Sigma_{\mathbf{r}\in\mathbf{\Pi}}(\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})a_{\mathbf{p}-\mathbf{r}}) \\
&= \Sigma_{\mathbf{p},\mathbf{q}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})a_{\mathbf{q}} \\
&= \omega(\overline{\mathbb{t}})a,
\end{aligned}
$$

as was claimed. □

We say that ω is the $\mathfrak{C}$-*connection* $\mathfrak{C}$-*form* of ∇.

Proposition 3.2. *For any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$ and any $\overline{\mathbb{t}} \in E^{D(\mathbf{1},\ldots,\mathbf{k})}$ we have*

$$(3.9)\quad \mathrm{q}_{(\zeta\circ\overline{\mathbb{t}},\mathbb{d})}(\overline{\mathbb{t}}(\mathbb{d})) = \overline{\mathbb{t}}(0) + \Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})\mathbb{d}_{\mathbf{p}}$$

Proof: Consider the mapping

$$(\mathbb{d}, \mathbb{d}') \in D(\mathbf{1}, \ldots, \mathbf{k}, \mathbf{1}, \ldots, \mathbf{k}) \mid \to \mathrm{p}_{(\zeta\circ\mathbb{t},\mathbb{d})}(\overline{\mathbb{t}}(0) + \Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})\mathbb{d}'_{\mathbf{p}}) \in E,$$

which coincides with $\nabla(\zeta \circ \overline{\mathbb{t}}, \overline{\mathbb{t}}(0))$ on the first $D(\mathbf{1}, \ldots, \mathbf{k})$ and with $\bar{\omega}(\overline{\mathbb{t}})$ on the second $D(\mathbf{1}, \ldots, \mathbf{k})$. Therefore the mapping

$$\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k}) \mid \to \mathrm{p}_{(\xi\circ\mathbb{t},\mathbb{d})}(\overline{\mathbb{t}}(0) + \Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\mathbf{p}}(\overline{\mathbb{t}})\mathbb{d}_{\mathbf{p}}) \in E$$

coincides with $\overline{\mathbb{t}}$, which implies the desired proposition. □

Now suppose that we are given a mapping $\varphi : M \to N$ of $\mathfrak{C}$-microlinear spaces and a $\mathfrak{C}$-vector bundle $\eta : F \to N$ endowed with a $\mathfrak{C}$-connection $\overline{\nabla}$, which shall be fixed throughout the rest of this section. Braided exterior differential calculus discussed in the previous section (particularly Propositions 2.2 and 2.3) can be generalized easily to braided covariant exterior differential calculus.

Proposition 3.3. *Given a differential n-$\mathfrak{C}$-form θ on M with values in η relative to φ, there exists a unique differential $(n+1)$-$\mathfrak{C}$- form on M with values in η relative to φ, to be denoted by $\mathbf{d}_{\bar{\nabla}}\theta$, such that for any $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) \in \mathbf{\Pi}^{n+1}$ and any $(\gamma, \underline{e}) \in \tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}} M$ with $\underline{e} = (e_1, \ldots, e_{n+1})$, we have*

$$\text{(3.10)}\quad \mathbf{d}_{\bar{\nabla}}\theta(\gamma)e_1 \ldots e_{n+1} = \Sigma_{i=1}^{n+1}(-1)^i \alpha_i \big(\theta\big(\gamma_0^i\big) - \mathrm{q}^{\bar{\nabla}}_{(\varphi\circ\gamma_i, e_i)}\big(\theta\big(\gamma_{e_i}^i\big)\big)\big) e_1 \ldots \hat{e_i} \ldots e_{n+1},$$

where γ_i is the tangent $\mathfrak{C}$-vector to M assigning $\gamma(0, \ldots, 0, d, 0, \ldots, 0)$ (d is positioned at the ith slot) to each $d \in D^{\mathbf{p}_i}$ and $\alpha_i = (n_{j=i+1}^{n+1} \delta^{\mathbf{p}_i, \mathbf{p}_j})(n_{k=1}^{i-1} \delta^{\mathbf{p}_k, \mathbf{p}_i})$.

Proof: By a direct generalization of Proposition 2.3. □

We call $\mathbf{d}_{\bar{\nabla}}\theta$ the *covariant exterior $\mathfrak{C}$-derivative* of θ.

4. INDUCED BRAIDED CONNECTIONS I

Let us define some induced $\mathfrak{C}$-connections. Let $\zeta : E \to M$ and $\eta : F \to M$ be $\mathfrak{C}$-vector bundles over the same base space M with $\mathfrak{C}$-connection ∇ and ∇' bestowed upon them. First we define an induced $\mathfrak{C}$-connection $\nabla \oplus \nabla'$ on the Whitney sum $\zeta \oplus \eta$ as follows:

(4.1) $(\nabla \oplus \nabla')(\mathbb{t}, v_\zeta \oplus v_\eta)(\mathbb{d}) = \nabla(\mathbb{t}, v_\zeta)(\mathbb{d}) \oplus \nabla'(\mathbb{t}, v_\eta)(\mathbb{d})$ for any $\mathbb{t} \in M^{D(\mathbf{1},\ldots,\mathbf{k})}$, any $v_\zeta \in E_{\mathbb{t}(0)}$, any $v_\eta \in F_{\mathbb{t}(0)}$ an any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$.

Proposition 4.1. *For any $\overline{\mathbb{t}_\zeta} \in E^{D(\mathbf{1},\ldots,\mathbf{k})}$ and any $\overline{\mathbb{t}_\eta} \in F^{D(\mathbf{1},\ldots,\mathbf{k})}$ with $\zeta^{D(\mathbf{1},\ldots,\mathbf{k})}(\overline{\mathbb{t}_\zeta}) = \eta^{D(\mathbf{1},\ldots,\mathbf{k})}(\overline{\mathbb{t}_\eta})$, we have*

$$\text{(4.2)}\quad \omega_{\zeta\oplus\eta}(\overline{\mathbb{t}_\zeta} \oplus \overline{\mathbb{t}_\eta}) = \omega_\zeta(\overline{\mathbb{t}_\zeta}) \oplus \omega_\eta(\overline{\mathbb{t}_\eta}),$$

where $\omega_{\zeta\oplus\eta}$, ω_ζ, and ω_η denote the $\mathfrak{C}$-connection $\mathfrak{C}$-forms of $\nabla \oplus \nabla'$, ∇ and, ∇', respectively.

Proof: Let $\mathbb{t} = \zeta^{D(\mathbf{1},\ldots,\mathbf{k})}(\overline{\mathbb{t}_\zeta}) = \eta^{D(\mathbf{1},\ldots,\mathbf{k})}(\overline{\mathbb{t}_\eta})$. For any $\mathbb{d} \in D(\mathbf{1}, \ldots, \mathbf{k})$, we have, by Proposition 3.2, that

$$\begin{aligned} \text{(4.3)}\quad & \mathrm{q}^{\nabla\oplus\nabla'}_{(\mathbb{t},\mathbb{d})}(\overline{\mathbb{t}_\zeta}(\mathbb{d}) \oplus \overline{\mathbb{t}_\eta}(\mathbb{d})) \\ &= (\overline{\mathbb{t}_\zeta}(0) + \Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}_\zeta})\mathbb{d}_p) \oplus (\overline{\mathbb{t}_\eta}(0) + \textstyle\sum_{\mathbf{p}\in\mathbf{\Pi}} \omega_{\eta,\mathbf{p}}(\overline{\mathbb{t}_\zeta})\mathbb{d}_{\mathbf{p}}) \\ &= (\overline{\mathbb{t}_\zeta}(0) \oplus \overline{\mathbb{t}_\eta}(0)) + \Sigma_{\mathbf{p}\in\mathbf{\Pi}}(\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}_\zeta}) \oplus \omega_{\eta,\mathbf{p}}(\overline{\mathbb{t}_\eta}))\mathbb{d}_{\mathbf{p}} \end{aligned}$$

Therefore the desired proposition obtains by Proposition 3.2 again. □

Corollary 4.2. *For any $\mu \in \mathrm{Sec}\,\zeta$ and any $\nu \in \mathrm{Sec}\,\eta$, we have*

(4.4) $\mathbf{d}_{\nabla\oplus\nabla'}(\mu\oplus\nu)=\mathbf{d}_{\nabla}\mu\oplus\mathbf{d}_{\nabla'}\nu$

We now difine an induced $\mathfrak{C}$-connection $\widehat{\nabla}$ on $\pi_{\mathscr{L}(\zeta,\eta)}$ as follows:

(4.5) $\widehat{\nabla}(\mathbb{t},\hat{v})(\mathbb{d})(v)=p^{\nabla}_{(\mathbb{t},\mathbb{d})}(\hat{v}(q^{\nabla}_{(\mathbb{t},\mathbb{d})}(v)))$
for any $\mathbb{t}\in M^{D(\mathbf{1},\ldots,\mathbf{k})}$, any $\mathbb{d}\in D(\mathbf{1},\ldots,\mathbf{k})$, any
$\hat{v}\in\mathscr{L}(\zeta,\eta)_{\mathbb{t}(0)}$ and any $v\in E_{\mathbb{t}(0)}$.

Proposition 4.3. *For any* $\hat{\mathbb{t}}\in\mathscr{L}(\zeta,\eta)^{D(\mathbf{1},\ldots,\mathbf{k})}$ *and any* $\overline{\mathbb{t}}\in E^{D(\mathbf{1},\ldots,\mathbf{k})}$ *with* $(\pi_{\mathscr{L}(\zeta,\eta)})^{D(\mathbf{1},\ldots,\mathbf{k})}(\hat{\mathbb{t}})=\zeta^{D(\mathbf{1},\ldots,\mathbf{k})}(\overline{\mathbb{t}})$, *we have*

(4.6) $\omega_\eta(\hat{\mathbb{t}}(\overline{\mathbb{t}}))=\Sigma_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathbf{\Pi}}\delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{p}}\delta^{\mathbf{p},\mathbf{q}}(\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}}))_{\mathbf{q}}(\overline{\mathbb{t}}(0)_{\mathbf{r}})+\hat{\mathbb{t}}(0)(\omega_\zeta(\overline{\mathbb{t}}))$,

where $\hat{\omega}$ *denote the* $\mathfrak{C}$*-connection form of* $\widehat{\nabla}$, *and* $\hat{\mathbb{t}}(\overline{\mathbb{t}})$ *denotes the mapping* $\mathbb{d}\in D(\mathbf{1},\ldots,\mathbf{k})\mid\rightarrow\hat{\mathbb{t}}(\mathbb{d})(\overline{\mathbb{t}}(\mathbb{d}))$. *In particular, if the braiding* Ψ *happens to be symmetric, then we have*

(4.7) $\omega_\eta(\hat{\mathbb{t}}(\overline{\mathbb{t}}))=\Sigma_{\mathbf{p},\mathbf{r}\in\mathbf{\Pi}}\delta^{\mathbf{p},\mathbf{r}}\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}})(\overline{\mathbb{t}}(0)_{\mathbf{r}})+\hat{\mathbb{t}}(0)(\omega_\zeta(\overline{\mathbb{t}}))$.

Proof: Let $\mathbb{t}=(\pi_{\mathscr{L}(\zeta,\eta)})^{D(\mathbf{1},\ldots,\mathbf{k})}(\hat{\mathbb{t}})=\zeta^{D(\mathbf{1},\ldots,\mathbf{k})}(\overline{\mathbb{t}})$. For any $\mathbb{d}\in D(\mathbf{1},\ldots,\mathbf{k})$, we have, by Proposition 3.2, that

(4.8)
$$\begin{aligned}
&\mathrm{q}^{\nabla'}_{(\mathbb{t},\mathbb{d})}(\hat{\mathbb{t}}(\mathbb{d})(\overline{\mathbb{t}}(\mathbb{d})))\\
&=\mathrm{q}^{\widehat{\nabla}}_{(\mathbb{t},\mathbb{d})}(\hat{\mathbb{t}}(\mathbb{d}))(\mathrm{q}^{\nabla}_{(\mathbb{t},\mathbb{d})}(\overline{\mathbb{t}}(\mathbb{d})))\\
&=(\hat{\mathbb{t}}(0)+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}})\mathbb{d}_{\mathbf{p}})(\overline{\mathbb{t}}(0)+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}})\mathbb{d}_{\mathbf{p}})\\
&=(\hat{\mathbb{t}}(0)+\Sigma_{\mathbf{p},\mathbf{q}\in\mathbf{\Pi}}\delta^{\mathbf{q},\mathbf{p}}\mathbb{d}_{\mathbf{p}}(\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}}))_{\mathbf{q}})(\overline{\mathbb{t}}(0)+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}})\mathbb{d}_{\mathbf{p}})\\
&=\hat{\mathbb{t}}(0)(\overline{\mathbb{t}}(0))+\Sigma_{\mathbf{p},\mathbf{q}\in\mathbf{\Pi}}\delta^{\mathbf{q},\mathbf{p}}\mathbb{d}_{\mathbf{p}}(\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}}))_{\mathbf{q}}(\overline{\mathbb{t}}(0))+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}}))\mathbb{d}_{\mathbf{p}}\\
&=\hat{\mathbb{t}}(0)(\overline{\mathbb{t}}(0))+\Sigma_{\mathbf{p},\mathbf{q}\in\mathbf{\Pi}}\delta^{\mathbf{q},\mathbf{p}}\mathbb{d}_{\mathbf{p}}(\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}}))_{\mathbf{q}}(\Sigma_{\mathbf{r}\in\mathbf{\Pi}}\overline{\mathbb{t}}(0)_{\mathbf{r}})+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}}))\mathbb{d}_{\mathbf{p}}\\
&=\hat{\mathbb{t}}(0)(\overline{\mathbb{t}}(0))+\Sigma_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathbf{\Pi}}\delta^{\mathbf{q},\mathbf{p}}\delta^{\mathbf{p},\mathbf{q}+\mathbf{r}}(\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}}))_{\mathbf{q}}(\overline{\mathbb{t}}(0)_{\mathbf{r}})\mathbb{d}_{\mathbf{p}}+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}}))\mathbb{d}_{\mathbf{p}}\\
&=\hat{\mathbb{t}}(0)(\overline{\mathbb{t}}(0))+\Sigma_{\mathbf{p}\in\mathbf{\Pi}}\{\Sigma_{\mathbf{q},\mathbf{r}\in\mathbf{\Pi}}\delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{p}}\delta^{\mathbf{p},\mathbf{q}}(\hat{\omega}_{\mathbf{p}}(\hat{\mathbb{t}}))_{\mathbf{q}}(\overline{\mathbb{t}}(0)_{\mathbf{r}})+\hat{\mathbb{t}}(0)(\omega_{\zeta,\mathbf{p}}(\overline{\mathbb{t}}))\}\mathbb{d}_{\mathbf{p}}
\end{aligned}$$

Therefore the desired proposition obtains by Proposition 3.2 again. □

Corollary 4.4. *For any* $\mu\in\mathrm{Sec}\,\zeta$ *and any* $\iota\in\mathrm{Sec}\,\pi_{\mathscr{L}(\zeta,\eta)}$, *we have*

(4.9) $d_{\nabla'}(\iota(\mu))=\Sigma_{\mathbf{p},\mathbf{q},\mathbf{r},\in\mathbf{\Pi}}\delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{p}}\delta^{\mathbf{p},\mathbf{q}}(d^{\mathbf{p}}_{\widehat{\nabla}}\iota)_{\mathbf{q}}(\mu_{\mathbf{r}})+\iota(d_{\nabla}\mu)$

In particular, if the braiding Ψ is symmetric, then we have

(4.10) $d_{\nabla'}(\iota(\mu))=\Sigma_{\mathbf{p},\mathbf{r},\in\mathbf{\Pi}}\delta^{\mathbf{p},\mathbf{r}}(d^{\mathbf{p}}_{\widehat{\nabla}}\iota)(\mu_r)+\iota(\mathbf{d}_{\nabla}\mu)$

If η is the trivial bundle $M \times \mathbb{R} \to M$ and the $\mathfrak{C}$-connection ∇' is trivial, then the $\mathfrak{C}$-connection $\widehat{\nabla}$ is usually denoted by ∇^*. If $\zeta = \eta$ and $\nabla = \nabla'$, then the $\mathfrak{C}$-connection $\widehat{\nabla}$ is usually denoted by $\tilde{\nabla}$.

5. CURVATURE

Let $\zeta : E \to M$ be a $\mathfrak{C}$-vector bundle endowed with a $\mathfrak{C}$-connection ∇, which shall be fixed throughout this section. The principal objective of this section is to introduce a sort of curvature abiding by the so-called second Bianchi identity. First let us introduce a preliminary version of curavature somewhat disobedient to the second Bianchi identity, from which our desired curvature naturally follows. The $\mathfrak{C}$-connection $\mathfrak{C}$-form ω is surely an element of $\Xi_1(E \xrightarrow{\zeta} M; \zeta)$, and its covariant exterior $\mathfrak{C}$-derivative $\mathbf{d}_\nabla \omega \in \Xi_2(E \xrightarrow{\zeta} M; \zeta)$ is called the *curvature $\mathfrak{C}$-form of the first kind* and denoted by Ω, for which we have

Proposition 5.1. *For any $\overline{\gamma} \in E^{D(\mathbf{p}) \times D(\mathbf{q})}$ and any $(d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$ with $\gamma = \zeta \circ \overline{\gamma}$, $t_1 = \gamma(\cdot, 0)$, $t_2 = \gamma(d_1, \cdot)$, $t_3 = \gamma(0, \cdot)$ and $t_4 = \gamma(\cdot, d_2)$, we have*

$$(5.1)\quad (\delta^{\mathbf{p},\mathbf{q}})^{-1}\Omega(\overline{\gamma})d_1 d_2 = \mathrm{q}_{(t_1,d_1)} \circ \mathrm{q}_{(t_2,d_2)}(\overline{\gamma}(d_1, d_2)) - \mathrm{q}_{(t_3,d_2)} \circ \mathrm{q}_{(t_4,d_1)}(\overline{\gamma}(d_1, d_2))$$

Proof: By the very definition of covariant exterior $\mathfrak{C}$-differentiation, we have

$$(5.2)\quad (\delta^{\mathbf{p},\mathbf{q}})^{-1}\Omega(\overline{\gamma})d_1 d_2 = \omega(\overline{\gamma}(\cdot, 0))d_1 + q_{(t_1,d_1)}(\omega(\overline{\gamma}(d_1, \cdot)))d_2 - \mathrm{q}_{(t_3,d_2)}(\omega(\overline{\gamma}(\cdot, d_2)))d_1 - \omega(\overline{\gamma}(0, \cdot))d_2$$

By Proposition 3.2 we have

$$(5.3)\quad \omega(\overline{\gamma}(\cdot, 0))d_1 = \mathrm{q}_{(t_1,d_1)}(\overline{\gamma}(d_1, 0)) - \overline{\gamma}(0, 0)$$

$$(5.4)\quad \mathrm{q}_{(t_1,d_1)}(\omega(\overline{\gamma}(d_1, \cdot)))d_2 = \mathrm{q}_{(t_1,d_1)}\{\mathrm{q}_{(t_2,d_2)}(\overline{\gamma}(d_1, d_2)) - \overline{\gamma}(d_1, 0)\} = \mathrm{q}_{(t_1,d_1)} \circ \mathrm{q}_{(t_2,d_2)}(\overline{\gamma}(d_1, d_2)) - \mathrm{q}_{(t_1,d_1)}(\overline{\gamma}(d_1, 0))$$

$$(5.5)\quad \mathrm{q}_{(t_3,d_2)}(\omega(\overline{\gamma}(\cdot, d_2)))d_1 = \mathrm{q}_{(t_3,d_2)}\{\mathrm{q}_{(t_4,d_1)}(\overline{\gamma}(d_1, d_2)) - \overline{\gamma}(0, d_2)\} = \mathrm{q}_{(t_3,d_2)} \circ \mathrm{q}_{(t_4,d_1)}(\overline{\gamma}(d_1, d_2)) - \mathrm{q}_{(t_3,d_2)}(\overline{\gamma}(0, d_2))$$

$$(5.6)\quad \omega(\overline{\gamma}(0, \cdot))d_2 = \mathrm{q}_{(t_3,d_2)}(\overline{\gamma}(0, d_2)) - \overline{\gamma}(0, 0)$$

Therefore the desired conclusion follows. □

Now we introduce another curvature $\mathfrak{C}$-form, to be called the *curvature $\mathfrak{C}$-form of the second kind* and to be denoted by $\tilde{\Omega}$, as follows:

(5.7) $\tilde{\Omega}(\overline{\gamma}) = \Omega(h(\overline{\gamma}))$ for any $\mathfrak{C}$-microsquare $\overline{\gamma}$ on E, where $h(\overline{\gamma})$ denotes the *horizontal component* of $\overline{\gamma}$ (cf. Moerdijk and Reyes, 1991, Chap.V. Section 6) in the sense that for any $(\mathbb{d}_1, \mathbb{d}_2) \in D(\mathbf{1}, \ldots, \mathbf{k})^2$,

(5.8) $h(\overline{\gamma})(\mathbb{d}_1, \mathbb{d}_2) = \mathrm{p}_{(\gamma(\mathbb{d}_1,\cdot),\mathbb{d}_2)} \circ \mathrm{p}_{(\gamma(\cdot,0),\mathbb{d}_1)}(\overline{\gamma}(0,0))$

with $\gamma = \zeta \circ \overline{\gamma}$. For the curvature $\mathfrak{C}$-form of the second kind, we have

Proposition 5.2. *Using the same notation as in Proposition* 5.1, *we have*

(5.9) $(\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\overline{\gamma})d_1 d_2 = \overline{\gamma}(0,0) - \mathrm{q}_{(t_3,d_2)} \circ \mathrm{q}_{(t_4,d_1)} \circ \mathrm{p}_{(t_2,d_2)} \circ \mathrm{p}_{(t_1,d_1)}(\overline{\gamma}(0,0))$,

so that $\tilde{\Omega}(\overline{\gamma})$ depends only on $\gamma = \zeta \circ \overline{\gamma}$ and $v = \overline{\gamma}(0,0)$, which enables us to regard $\tilde{\Omega}$ as a function from $\mathbf{T}^2 M$ to $\mathscr{L}(\zeta)$ in the sense that $\tilde{\Omega}(\gamma)(v) = \tilde{\Omega}(\overline{\gamma})$.

Proof: Simply put $h(\overline{\gamma})$ in place of $\overline{\gamma}$ in Proposition 5.1. □

We now reckon $\tilde{\Omega}$ as a function from $M^{D(\mathbf{1},\ldots,\mathbf{k})^2}$ to $\mathscr{L}(\zeta)$ in the canonical way, for which we have

Proposition 5.3. *The function $\tilde{\Omega} : M^{D(\mathbf{1},\ldots,\mathbf{k})^2} \to \mathscr{L}(\zeta)$ is a differential* 2-*$\mathfrak{C}$-form on M with values in $\pi_{\mathscr{L}(\zeta)}$. I.e., $\tilde{\Omega} \in \Xi_2(M; \pi_{\mathscr{L}(\zeta)})$.*

Proof: We define a function $\mathfrak{h} : M^{D(\mathbf{1},\ldots,\mathbf{k})^2} \underset{M}{\times} E \to E^{D(\mathbf{1},\ldots,\mathbf{k})^2}$ as follows:

(5.10) $\mathfrak{h}(\gamma, v)(\mathbb{d}_1, \mathbb{d}_2) = \mathrm{p}_{(\gamma(\mathbb{d}_1,\cdot),\mathbb{d}_2)} \circ \mathrm{p}_{(\gamma(\cdot,0),\mathbb{d}_1)}(v)$ for any $(\gamma, v) \in M^{D(\mathbf{1},\ldots,\mathbf{k})^2} \underset{M}{\times} E$ and any $(\mathbb{d}_1, \mathbb{d}_2) \in D(\mathbf{1}, \ldots, \mathbf{k})^2$.

Then it is easy to see that

(5.11) $\mathfrak{h}(\gamma \dot{}_i a, v) = \mathfrak{h}(\gamma, v) \dot{}_i a$ for any $a \in \mathbb{R}(i = 1, 2)$.

Since $\tilde{\Omega}(\gamma)(v) = \Omega(b(\gamma, v))$ and Ω satisfies (2.6), $\tilde{\Omega}$ also satisfies (2.6). Now we use the same notation as in Proposition 5.1 and 5.2. To show that $\tilde{\Omega}$ satisfies (2.5), we Let $v_0 = v$ and define v_1 and v_2 in order as follows:

(5.12) $v_1 = \mathrm{q}_{(t_3,d_2)} \circ \mathrm{q}_{(t_4,d_1)} \circ \mathrm{p}_{(t_2,d_2)} \circ \mathrm{p}_{(t_1,d_1)}(v_0)$
(5.13) $v_2 = \mathrm{q}_{(t_1,d_1)} \circ \mathrm{q}_{(t_2,d_2)} \circ \mathrm{p}_{(t_4,d_1)} \circ \mathrm{p}_{(t_3,d_2)}(v_1)$

On the one hand it follows directly from (5.12) and (5.13) that

(5.14) $v_2 = v_0$

On the other hand we can calculate v_1 and v_2 in order by making use of Proposition 5.2:

$$(5.15)\quad v_1 = v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma)(v_0)d_1d_2$$

$$\begin{aligned}(5.16)\quad v_2 &= v_1 - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}(\Sigma(\gamma))(v_1)d_2d_1 \\ &= v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma)(v_0)d_1d_2 \\ &\quad - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}(\Sigma(\gamma))(v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma)(v_0)d_1d_2)d_2d_1 \quad [(5.15)] \\ &= v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma)(v_0)d_1d_2 - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}(\Sigma(\gamma))(v_0)d_2d_1 \\ &= v_0 - \tilde{\Omega}(\gamma)(v_0)d_2d_1 - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}(\Sigma(\gamma))(v_0)d_2d_1 \\ &\quad [\text{since } d_1d_2 = \delta^{\mathbf{p},\mathbf{q}}d_2d_1]\end{aligned}$$

It follows from (5.14) and (5.16) that

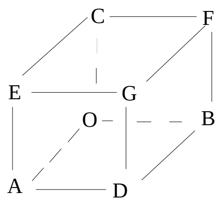
$$(5.17)\quad \delta^{\mathbf{q},\mathbf{p}}\tilde{\Omega}(\gamma)(v_0) + \tilde{\Omega}(\Sigma(\gamma))(v_0) = 0,$$

which means that $\tilde{\Omega}$ satisfies (2.5). □

The above proof is a prototype of the proof of Theorem 5.5 to come.

Now we give a braided, cubical version of Kock's simplicial and combinatorial Bianchi identity (Kock, 1996, Theorem 2).

Theorem 5.4. *Let* $\gamma \in M^{D(\mathbf{p})\times D(\mathbf{q})\times D(\mathbf{r})}$. Let $(d_1, d_2, d_3) \in D(\mathbf{p}) \times D(\mathbf{q}) \times D(\mathbf{r})$. *We denote points* $\gamma(0,0,0)$, $\gamma(d_1,0,0)$, $\gamma(0,d_2,0)$, $\gamma(0,0,d_3)$, $\gamma(d_1,d_2,0)$, $\gamma(d_1,0,d_3)$, $\gamma(0,d_2,d_3)$, *and* $\gamma(d_1,d_2,d_3)$ *by* O, A, B, C, D, E, F, *and* G, *respectively. These eight points are depicted figuratively as the eight vertices of a cube:*



Then we have

$$(5.18)\quad P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GFBD}} \circ R_{\mathrm{GECF}} \circ R_{\mathrm{GDAE}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}} \circ R_{\mathrm{OCEA}} \circ R_{\mathrm{OBFC}} \circ R_{\mathrm{OADB}} = \mathrm{id}_{\mathrm{O}},$$

where

(5.19) *for any adjacent vertices* X, Y *of the cube,* P_{XY} *denotes the parallel transport from* X *to* Y *along the line connecting* X *and* Y *(e.g.,* P_{OA} *and* P_{AO} *denote* $\mathrm{p}_{(\gamma(\cdot,0),d_1)}$ *and* $\mathrm{q}_{(\gamma(\cdot,0),d_1)}$*, respectively),* (5.20) *for any four vertices* X, Y, Z, W *of the cube rounding one of the six facial squares of the cube,* R_{XYZW} *denotes* $P_{\mathrm{WX}} \circ P_{\mathrm{ZW}} \circ P_{\mathrm{YZ}} \circ P_{\mathrm{XY}}$ *(e.g.,* R_{OADB} *denotes* $\mathrm{q}_{(\gamma(0,\cdot,\ 0),d_2)} \circ q_{(\gamma(\ \cdot,\ d_2,\mathrm{O}),d_1)} \circ \mathrm{P}_{(\gamma(d_1,\cdot,\mathrm{O}),d_2)} \circ \mathrm{P}_{(\gamma(\ \cdot\ ,\mathrm{O},\mathrm{O}),d_1))}$ *and*

(5.21) id_{O} *is the identity transfotmation of* E_{O}.

Proof: Write over (5.18) exclusively in terms of P_{XY}'s, and write off all consective $P_{\mathrm{XY}} \circ P_{\mathrm{YX}}$'s □

The above theorem gives rise to the following form of the second Bianchi identity in our braided context.

Theorem 5.5. *We have*

(5.22) $\mathbf{d}_{\tilde{\nabla}}\tilde{\Omega} = 0,$

where $\mathbf{d}_{\tilde{\nabla}}$ *is the covariant exterior* $\mathfrak{C}$*-differentiation with respect to the induced* $\mathfrak{C}$*-connection* $\tilde{\nabla}$ *on* $\pi_{\mathscr{L}(\xi)}$*, and recall that* $\tilde{\Omega} \in \Xi_2(M;\pi_{\mathscr{L}(\xi)})$*, as was explained in Proposition* 5.3.

Proof: The proof is carried out by the same method as in Proposition 5.3. Let $\gamma,\ d_1,\ d_2,\ d_3$, O, A, B, C, D, E, F, and G be as in Theorem 5.4. Given $v_0 \in E_{\gamma(0,\ 0,\ 0)}$, we define $v_i \in E_{\gamma(0,\ 0,\ 0)}$ $(i = 1,\ 2,\ 3,\ 4,\ 5,\ 6)$ in order as follows:

(5.23) $v_1 = R_{\mathrm{OADB}}(v_0)$

(5.24) $v_2 = R_{\mathrm{OBFC}}(v_1)$

(5.25) $v_3 = R_{\mathrm{OCEA}}(v_2)$

(5.26)
$$\begin{aligned} v_4 &= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GDAE}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_3) \\ &= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ P_{\mathrm{OA}}(v_3) \end{aligned}$$

(5.27)
$$\begin{aligned} v_5 &= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GECF}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_4) \\ &= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ P_{\mathrm{EA}} \circ R_{\mathrm{ECFG}} \circ P_{\mathrm{AE}} \circ R_{\mathrm{ADGE}} \circ P_{\mathrm{OA}}(v_4) \\ &= P_{\mathrm{AO}} \circ P_{\mathrm{EA}} \circ R_{\mathrm{EGDA}} \circ R_{\mathrm{ECFG}} \circ R_{\mathrm{EADG}} \circ P_{\mathrm{AE}} \circ P_{\mathrm{OA}}(v_4) \\ &= R_{\mathrm{OCEA}} \circ P_{\mathrm{CO}} \circ P_{\mathrm{EC}} \circ R_{\mathrm{EGDA}} \circ R_{\mathrm{ECFG}} \circ R_{\mathrm{EADG}} \circ P_{\mathrm{CE}} \circ P_{\mathrm{OC}} \circ \\ &\quad R_{\mathrm{OAEC}}(v_4) \\ &= R_{\mathrm{OCEA}} \circ P_{\mathrm{CO}} \circ P_{\mathrm{EC}} \circ R_{\mathrm{EGDA}} \circ P_{\mathrm{CE}} \circ R_{\mathrm{CFGE}} \circ P_{\mathrm{EC}} \circ R_{\mathrm{EADG}} \circ P_{\mathrm{CE}} \circ \\ &\quad P_{\mathrm{OC}} \circ R_{\mathrm{OAEC}}(v_4) \end{aligned}$$

(5.28)
$$\begin{aligned} v_6 &= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ P_{\mathrm{GD}} \circ R_{\mathrm{GFBD}} \circ P_{\mathrm{DG}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_5) \\ &= P_{\mathrm{AO}} \circ P_{\mathrm{DA}} \circ R_{\mathrm{DGFB}} \circ P_{\mathrm{AD}} \circ P_{\mathrm{OA}}(v_5) \\ &= R_{\mathrm{OBDA}} \circ P_{\mathrm{BO}} \circ R_{\mathrm{BDGF}} \circ P_{\mathrm{OB}} \circ R_{\mathrm{OADB}}(v_5) \end{aligned}$$

Now we calculate $v_i (i = 1, \ldots, 6)$ in order. It follows directly from Proposition 5.2 that

(5.29) $v_1 = v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))(v_0)d_1 d_2$

The calculations of v_2 and v_3 are similar, so we present details of the former calculation but simply register the result of the latter calculation, safely leaving its details to the reader.

$$(5.30)\quad \begin{aligned} v_2 &= v_1 - (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_1)d_2d_3 \quad [\text{Proposition 5.2}] \\ &= v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 - (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot)) \\ &\qquad \times (v_0 - \delta^{\mathbf{p},\mathbf{q}}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2)d_2d_3 \quad [(5.29)] \\ &= v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 - (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \end{aligned}$$

$$(5.31)\quad \begin{aligned} v_3 &= v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 - (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \\ &\quad + (\delta^{\mathbf{p},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)d_1d_3 \end{aligned}$$

The three calculations of v_4, v_5, and v_6 are similar, so we present their details only in case of the first, leaving details of the other two calculations to the reader.

$$(5.32)\quad \begin{aligned} v_4 &= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}} \circ P_{\mathrm{OA}}(v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 \\ &\quad - (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \\ &\quad + (\delta^{\mathbf{p},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)d_1d_3) \quad [(5.31)] \\ &= P_{\mathrm{AO}} \circ R_{\mathrm{AEGD}}(P_{\mathrm{OA}}(v_0) - (\delta^{\mathbf{p},\mathbf{q}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0))d_1d_2 \\ &\quad - (\delta^{\mathbf{q},\mathbf{r}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0))d_2d_3 \\ &\quad + (\delta^{\mathbf{p},\mathbf{r}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0))d_1d_3) \\ &= P_{\mathrm{AO}}(P_{\mathrm{OA}}(v_0) - (\delta^{\mathbf{p},\mathbf{q},})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0))d_1d_2 \\ &\quad - (\delta^{\mathbf{q},\mathbf{r}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0))d_2d_3 \\ &\quad + (\delta^{\mathbf{p},\mathbf{r}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0))d_1d_3 \\ &\quad + (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))\,(P_{\mathrm{OA}}(v_0) \\ &\quad - (\delta^{\mathbf{q},\mathbf{p}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0))d_1d_2 \\ &\quad - (\delta^{\mathbf{r},\mathbf{q}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0))d_2d_3 \\ &\quad + (\delta^{\mathbf{r},\mathbf{p}})^{-1}P_{\mathrm{OA}}(\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)))d_1d_3)d_2d_3 \\ &\qquad [\text{Propositions 5.2 and 5.3}] \\ &= v_0 - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0))d_1d_2 \\ &\qquad - (\delta^{\mathbf{r},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 + (\delta^{\mathbf{r},\mathbf{p}})^{-1}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)d_1d_3 \\ &\qquad + (\delta^{\mathbf{q},\mathbf{r}})^{-1}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))\,(P_{\mathrm{OA}}(v_0)))d_2d_3 \end{aligned}$$

$$(5.33)\quad \begin{aligned} v_5 &= v_0 - (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 - (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \\ &\qquad + (\delta^{\mathbf{p},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0))d_1d_3 \\ &\qquad + (\delta^{\mathbf{q},\mathbf{r}})^{-1}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))\,(P_{\mathrm{OA}}(v_0)))d_2d_3 \\ &\qquad + (\delta^{\mathbf{p},\mathbf{q}})^{-1}P_{\mathrm{CO}}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))\,(P_{\mathrm{OC}}(v_0))d_1d_2 \end{aligned}$$

$$(5.34)\quad \begin{aligned} v_6 = v_0 &- (\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0))d_1d_2 \\ &- (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \\ &+ (\delta^{\mathbf{p},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)d_1d_3 \\ &+ (\delta^{\mathbf{q},\mathbf{r}})^{-1}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))\,(P_{\mathrm{OA}}(v_0)))d_2d_3 \\ &+ (\delta^{\mathbf{p},\mathbf{q}})^{-1}P_{\mathrm{CO}}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))\,(P_{\mathrm{OC}}(v_0)))d_1d_2 \\ &- (\delta^{\mathbf{p},\mathbf{r}})^{-1}P_{\mathrm{BO}}(\tilde{\Omega}(\gamma(\cdot,d_2,\cdot))\,(P_{\mathrm{OB}}(v_0)))d_1d_3 \end{aligned}$$

It should be the case by Theorem 5.4 that $v_6 = v_0$. Therefore

$$(5.35)\quad \begin{aligned} &(\delta^{\mathbf{p},\mathbf{q}})^{-1}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 + (\delta^{\mathbf{q},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \\ &- (\delta^{\mathbf{p},\mathbf{r}})^{-1}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)d_1d_3 \\ &- (\delta^{\mathbf{q},\mathbf{r}})^{-1}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))\,(P_{\mathrm{OA}}(v_0)))d_2d_3 \\ &- (\delta^{\mathbf{p},\mathbf{q}})^{-1}P_{\mathrm{CO}}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3,))\,(P_{\mathrm{OC}}(v_0)))d_1d_2 \\ &+ (\delta^{\mathbf{p},\mathbf{r}})^{-1}P_{\mathrm{BO}}(\tilde{\Omega}(\gamma(\cdot,d_2,\cdot))\,(P_{\mathrm{OB}}(v_0))))d_1d_3 = 0 \end{aligned}$$

By multiplying $\delta^{\mathbf{p},\mathbf{q}}\,\delta^{\mathbf{p},\mathbf{r}}\,\delta^{\mathbf{q},\mathbf{r}}$ upon (5.35), we have

$$(5.36)\quad \begin{aligned} &\delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{r}}\tilde{\Omega}(\gamma(\cdot,\cdot,0))\,(v_0)d_1d_2 + \delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{p},\mathbf{r}}\tilde{\Omega}(\gamma(0,\cdot,\cdot))\,(v_0)d_2d_3 \\ &- \delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{q},\mathbf{r}}\tilde{\Omega}(\gamma(\cdot,0,\cdot))\,(v_0)d_1d_3 \\ &- \delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{p},\mathbf{r}}P_{\mathrm{AO}}(\tilde{\Omega}(\gamma(d_1,\cdot,\cdot))\,(P_{\mathrm{OA}}(v_0)))d_2d_3 \\ &- \delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{r}}P_{\mathrm{CO}}(\tilde{\Omega}(\gamma(\cdot,\cdot,d_3))\,(P_{\mathrm{OC}}(v_0)))d_1d_2 \\ &+ \delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{q},\mathbf{r}}P_{\mathrm{BO}}(\tilde{\Omega}(\gamma(\cdot,d_2,\cdot))\,(P_{\mathrm{OB}}(v_0))))d_1d_3 = 0 \end{aligned}$$

Since $v_0 \in E_{\gamma(0,0,0)}$ was chosen arbitrarily, the proof is complete. □

6. INDUCED BRAIDED CONNECTIONS II

We calculate the curvature of the second kind of the induced $\mathfrak{C}$-connections dealt with in Section 4. Let $\zeta : E \to M$ and $\eta : F \to M$ be $\mathfrak{C}$-vector bundles over the same base space M embellished with $\mathfrak{C}$-connections ∇ and ∇', as in that section.

Proposition 6.1. *For any $\gamma \in M^{D(\mathbf{1},\ldots,\mathbf{k})^2}$ we have*

$$(6.1)\quad \tilde{\Omega}_{\zeta\oplus\eta}(\gamma) = \tilde{\Omega}_{\zeta}(\gamma) \oplus \tilde{\Omega}_{\eta}(\gamma),$$

where $\tilde{\Omega}_{\zeta\oplus\eta}$, $\tilde{\Omega}_{\zeta}$, and $\tilde{\Omega}_{\eta}$ denote the curvature forms of the second kind of $\mathfrak{C}$-connections $\nabla \oplus \nabla'$, ∇, and ∇', respectively.

Proof: Let $v \oplus v' \in (E \oplus F)_{\gamma(0,0)}$. We assume that $\gamma \in M^{D(\mathbf{p})\times D(\mathbf{q})}$. Let $d_1 \in D(\mathbf{p})$ and $d_2 \in D(\mathbf{q})$. Let $t_1 = \gamma(\cdot, 0)$, $t_2 = \gamma(d_1, \cdot)$, $t_3 = \gamma(0, \cdot)$ and $t_4 = \gamma(\cdot, d_2)$.

By Proposition 5.2, we have

$$(6.2)\quad (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}_{\zeta\oplus\eta}(\gamma)(v\oplus v')d_1d_2$$
$$= (v\oplus v') - \mathrm{q}^{\nabla\oplus\nabla'}_{(t_3,d_2)}\circ \mathrm{q}^{\nabla\oplus\nabla'}_{(t_4,d_1)}\circ \mathrm{p}^{\nabla\oplus\nabla'}_{(t_2,d_2)}\circ \mathrm{p}^{\nabla\oplus\nabla'}_{(t_1,d_1)}(v\oplus v')$$
$$= \left\{v - \mathrm{q}^{\nabla}_{(t_3,d_2)}\circ \mathrm{q}^{\nabla}_{(t_4,d_1)}\circ \mathrm{p}^{\nabla}_{(t_2,d_2)}\circ \mathrm{p}^{\nabla}_{(t_1,d_1)}(v)\right\}$$
$$\oplus \left\{v' - \mathrm{q}^{\nabla'}_{(t_3,d_2)}\circ \mathrm{q}^{\nabla'}_{(t_4,d_1)}\circ \mathrm{p}^{\nabla'}_{(t_2,d_2)}\circ \mathrm{p}^{\nabla'}_{(t_1,d_1)}(v')\right\}$$
$$= (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}_{\zeta}(\gamma)(v)d_1d_2 \oplus (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}_{\eta}(\gamma')(v')d_1d_2$$
$$= (\delta^{\mathbf{q},\mathbf{p}})^{-1}\{\tilde{\Omega}_{\zeta}(\gamma)(v) \oplus \tilde{\Omega}_{\eta}(\gamma')(v')\}d_1d_2$$

Therefore the desired proposition obtains. □

Proposition 6.2. *Let* $\gamma \in M^{D(\mathbf{p})\times D(\mathbf{q})}$ *and* $\hat{v} \in \mathscr{L}(\zeta,\eta)_{\gamma(0,0)}$. *Then we have*

$$(6.3)\quad \hat{\tilde{\Omega}}(\gamma)(\hat{v}) = -\hat{v}\circ\tilde{\Omega}_{\zeta}(\gamma) + \Sigma_{\mathbf{r},\mathbf{s}\in\mathbf{\Pi}}\,\delta^{\Gamma,\mathbf{p}+\mathbf{q}}\,\delta^{\mathbf{p}+\mathbf{q},\mathbf{r}+\mathbf{s}}(\tilde{\Omega}_{\eta}(\gamma))_{\mathbf{r}}\circ\hat{v}_{\mathbf{s}}$$

where $\hat{\tilde{\Omega}}$, $\tilde{\Omega}_{\zeta}$, *and* $\tilde{\Omega}_{\eta}$ *denote the curvature forms of the second kind of* $\mathfrak{C}$-*connections* $\hat{\nabla}$, ∇, *and* ∇', *respectively. In particular, if the braiding* Ψ *happens to be symmetric, then we have*

$$(6.4)\quad \hat{\tilde{\Omega}}(\gamma)(\hat{v}) = -\hat{v}\circ\tilde{\Omega}_{\zeta}(\gamma) + \Sigma_{\mathbf{s}\in\mathbf{\Pi}}\delta^{\mathbf{p}+\mathbf{q},s}\tilde{\Omega}_{\eta}(\gamma)\circ\hat{v}_{\mathbf{s}}$$

Proof: Let $d_1 \in D(\mathbf{p})$ and $d_2 \in D(\mathbf{q})$. By Proposition 5.2 we have

$$(6.5)\quad (\delta^{\mathbf{q},\mathbf{p}})^{-1}\hat{\tilde{\Omega}}(\gamma)(\hat{v})d_1d_2$$
$$= \hat{v} - \mathrm{q}^{\hat{\nabla}}_{(t_3,d_2)}\circ \mathrm{q}^{\hat{\nabla}}_{(t_4,d_1)}\circ \mathrm{p}^{\hat{\nabla}}_{(t_2,d_2)}\circ \mathrm{p}^{\hat{\nabla}}_{(t_1,d_1)}(\hat{v})$$
$$= \hat{v} - \mathrm{q}^{\nabla}_{(t_3,d_2)}\circ \mathrm{q}^{\nabla}_{(t_4,d_1)}\circ \mathrm{p}^{\nabla}_{(t_2,d_2)}\circ \mathrm{p}^{\nabla}_{(t_1,d_1)}\circ\hat{v}\circ$$
$$\mathrm{q}^{\nabla}_{(t_1,d_1)}\circ \mathrm{q}^{\nabla}_{(t_2,d_2)}\circ \mathrm{p}^{\nabla}_{(t_4,d_1)}\circ \mathrm{p}^{\nabla}_{(t_3,d_2)}$$
$$= \hat{v} - \left\{\mathrm{id}_{F_{\gamma(0,0)}} - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}_{\eta}(\gamma)d_1d_2\right\}\circ\hat{v}\circ\mathrm{id}_{E_{\gamma(0,0)}} + (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}_{\zeta}(\gamma)d_1d_2\right\}$$
$$= \hat{v} - \left\{\mathrm{id}_{F_{\gamma(0,0)}} - (\delta^{\mathbf{q},\mathbf{p}})^{-1}\Sigma_{\Gamma\in\mathbf{\Pi}}\delta^{\mathbf{r},\mathbf{p}+\mathbf{q}}d_1d_2(\tilde{\Omega}_{\eta}(\gamma))_{\mathbf{r}}\right\}\circ\{\Sigma_{\mathbf{s}\in\mathbf{\Pi}}\hat{v}_{\mathbf{s}}\}\circ$$
$$\{\mathrm{id}_{E_{\gamma(0,0)}} + (\delta^{\mathbf{q},\mathbf{p}})^{-1}\tilde{\Omega}_{\zeta}(\gamma)d_1d_2\}$$
$$= -(\delta^{\mathbf{q},\mathbf{p}})^{-1}\hat{v}\circ\tilde{\Omega}_{\zeta}(\gamma)d_1d_2$$
$$+ (\delta^{\mathbf{q},\mathbf{p}})^{-1}\Sigma_{\mathrm{r},\mathbf{s}\in\mathbf{\Pi}}\delta^{\Gamma,\mathbf{p}+\mathbf{q}}\delta^{\mathbf{p}+\mathbf{q},\mathbf{r}+\mathbf{s}}(\tilde{\Omega}_{\eta}(\gamma))_{\mathbf{r}}\circ\hat{v}_{\mathbf{s}}d_1d_2$$
$$= (\delta^{\mathbf{q},\mathbf{p}})^{-1}\{-\hat{v}\circ\tilde{\Omega}_{\zeta}(\gamma) + \Sigma_{\mathbf{r},\mathbf{s}\in\mathbf{\Pi}}\delta^{\mathbf{r},\mathbf{p}+\mathbf{q}}\,\delta^{\mathbf{p}+\mathbf{q},\mathbf{r}+\mathbf{s}}(\tilde{\Omega}_{\eta}(\gamma))_{\mathbf{r}}\circ\hat{v}_{\mathbf{s}}\}d_1d_2$$

Therefore the desired proposition obtains. □

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